One interesting consequence of the negative sign in the metric of special relativity is as follows:

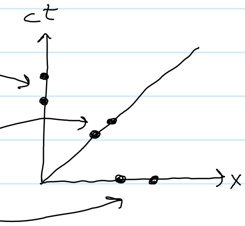
In 3D $\quad ds^2 = dx_i dx^i = dx^2 + dy^2 + dz^2 \geq 0$ and only 0 when $dx^i = 0$.

In SR $\quad ds^2 = dx_\mu dx^\mu = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ can be $+, -, 0$ and more interestingly, can be 0 even when $dx^\mu \neq 0$!

Now ds^2 is an invariant so it is the same in any reference frame. That means if it is negative, it will be negative to all observers (same for positive and 0).

But notice that the sign is controlled by how the coordinate velocity compares to c.

If for example we take $dy = dz = 0$ then for
$$\underbrace{\tfrac{dx}{dt}}_{\text{consider } dx=0} < c \quad ds^2 < 0 \quad \text{timelike}$$
$$\tfrac{dx}{dt} = c \quad ds^2 = 0 \quad \text{lightlike}$$
$$\tfrac{dx}{dt} > c \quad ds^2 > 0 \quad \text{spacelike}$$



Also recall: $P_\mu P^\mu = \underbrace{-\frac{E^2}{c^2} + p^2}_{} = -m^2c^2$ or $\boxed{E^2 - p^2c^2 = m^2c^4}$

$$P_\mu P^\mu \begin{cases} < 0 & \text{timelike} \Rightarrow m^2 > 0 & \text{massive} \\ = 0 & \text{lightlike} \Rightarrow m^2 = 0 & \text{massless} \\ > 0 & \text{spacelike} \Rightarrow m^2 < 0 & \text{tachyonic} \end{cases}$$

In nonrelativistic physics we often the spatial derivative $\vec{\nabla}$ which operates on fields, e.g. $\underbrace{\vec{\nabla}\phi}_{\text{gradient}}, \underbrace{\vec{\nabla}\cdot\vec{E}}_{\text{divergence}}$, and time derivative $\frac{\partial}{\partial t}$ which operates on both fields, e.g. $\frac{\partial \vec{B}}{\partial t}$, and particle degrees of freedom, e.g. $\frac{d\vec{r}}{dt}$.

When we move to special relativity, one might naively think that we just combine these to make a 4-derivative. We do, but this is only part of the story.

First: $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t} \; \frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z}\right)$ is the 4-derivative that acts on fields.

Just as the index placement suggests, this is in fact a dual vector, i.e. $\partial_\mu \to \partial_{\mu'} = \Lambda^\mu{}_{\mu'}\partial_\mu$

$$\left.\begin{array}{l}\text{Hence the 4-gradient } \partial_\mu \phi \text{ is a dual vector} \\ \text{and the 4-divergence } \partial_\mu A^\mu \text{ is a scalar}\end{array}\right\} \text{similar to 3D}$$

We will encounter this type of derivative a lot when we work in terms of fields, particularly when constructing Lagrangians and equations of motion.

But that is not the end of the story...

In special relativity, <u>anything</u> that was a vector in 3D must become a vector in 4D.

This includes coordinate related things like $d\vec{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$, $\vec{v} = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$ and also more abstract things like $\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$, $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$.

To promote $\vec{E}$ + $\vec{B}$ to 4D we actually have two options:

1) Combine $\vec{E}$ + $\vec{B}$ into an antisymmetric (0,2) tensor $\underset{\text{RC}}{F_{\mu\nu}} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$

Then $F_{\mu\nu} \to F_{\mu'\nu'} = \Lambda^{\alpha}{}_{\mu'} \Lambda^{\beta}{}_{\nu'} F_{\alpha\beta}$

or

2) Use the scalar and vector potentials $\vec{E} = \vec{\nabla}\phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}$ to form a vector $A^{\mu} = \begin{pmatrix} -\phi \\ \vec{A} \end{pmatrix}$

$\Downarrow$

Then $A^{\mu} \to A^{\mu'} = \Lambda^{\mu'}{}_{\nu} A^{\nu}$

$A_{\mu} = (\phi, \vec{A}) = (\phi, A_x, A_y, A_z)$

$= (A_0\ A_1\ A_2\ A_3)$

These are equivalent if we say $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$= (\partial_0, \partial_1, \partial_2, \partial_3)$

Example: $F_{01} = \partial_0 A_1 - \partial_1 A_0$

$= \frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x}$

$= -E_x$

4.

a) Consider a scattering process where $A + B \rightarrow C + D + E + F$ where particle A begins with (relativistic) energy E_A and particle B begins at rest. Consider all particle masses known, i.e. $m_A, m_B, m_C, m_D, m_E, m_F$. Calculate the minimum energy E_A required for this process to occur in terms of $m_A, m_B, m_C, m_D, m_E, m_F$. You should be able to generalize your result to an arbitrary number of final particles.

Since B begins at rest $\vec{p}_B = 0$. We want to find (in terms of $m_A, m_B, m_C, m_D, m_E, m_F$) the minimum E_A needed for this process to occur.

Important: In any process in which new particles are created, then in the center of momentum frame (where $\vec{p}_{tot} = 0$) the minimum total energy the new particles must have is generally their rest energy. If more than enough total energy is present, the final particles will also be moving, but again in the minimum energy case they are created at rest in the c.o.m. frame.

So in the Lab frame we have: $P^{\mu}_{tot,lab} = P_A^{\mu} + P_B^{\mu} = \begin{pmatrix} E_A/c \\ \vec{p}_A \end{pmatrix} + \begin{pmatrix} m_B c \\ \vec{0} \end{pmatrix} = \begin{pmatrix} E_A/c + m_B c \\ \vec{p}_A \end{pmatrix}$

Meanwhile in the c.o.m. frame: $P^{\mu}_{tot,com} = P_C^{\mu} + P_D^{\mu} + P_E^{\mu} + P_F^{\mu} = \begin{pmatrix} m_C c \\ \vec{0} \end{pmatrix} + \begin{pmatrix} m_D c \\ \vec{0} \end{pmatrix} + \begin{pmatrix} m_E c \\ \vec{0} \end{pmatrix} + \begin{pmatrix} m_F c \\ \vec{0} \end{pmatrix}$

Note: The minimum energy required leaves the outgoing particles at rest.

$= \begin{pmatrix} m_C c + m_D c + m_E c + m_F c \\ \vec{0} \end{pmatrix}$

Now we cannot say $P^{\mu}_{tot,lab} = P^{\mu}_{tot,com}$ but we can say $\underbrace{(P_{tot\,\mu} P_{tot}^{\mu})_{lab} = (P_{tot\,\mu} P_{tot}^{\mu})_{com}}$

Since these are invariants, i.e. the same in any frame.

Then: $(P_{A\mu} + P_{B\mu})(P_A^{\mu} + P_B^{\mu})_{lab} = \left(-\frac{E_A}{c} - m_B c \quad \vec{p}_A\right)\begin{pmatrix} E_A/c + m_B c \\ \vec{p}_A \end{pmatrix} = -\frac{E_A^2}{c^2} - m_B^2 c^2 - 2E_A m_B + p_A^2$

$= -m_A^2 c^2 - m_B^2 c^2 - 2E_A m_B$

Recall: $\frac{E_A^2}{c^2} - p_A^2 = m_A^2 c^2$ always!

And: $(P_{C\mu} + P_{D\mu} + P_{E\mu} + P_{F\mu})(P_C^{\mu} + P_D^{\mu} + P_E^{\mu} + P_F^{\mu})_{com} = -(m_C c + m_D c + m_E c + m_F c)^2$

Finally: $-m_A^2 c^2 - m_B^2 c^2 - 2E_A m_B = -(m_C c + m_D c + m_E c + m_F c)^2$

$$E_A = \frac{1}{2m_B}\left[\underbrace{(m_C c + m_D c + m_E c + m_F c)^2}_{\text{This can be generalized to any number of final particles.}} - m_A^2 c^2 - m_B^2 c^2\right]$$